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Two degrees of freedom quasi-bi-Hamiltonian systems

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Abstract. Starting from the classical example of the Hénon–Heiles integrable Hamiltonian system, we show that it admits a slightly different formulation from the classical bi-Hamiltonian system. We introduce the general notion of a quasi-bi-Hamiltonian system (QBHS) and study some of its basic properties only in the case of two degrees of freedom; in particular, we give the general form of this structure in the Nijenhuis coordinates which are constructed using the eigenvalues of the Nijenhuis operator of the system and show how they serve to separate the Hamilton–Jacobi equation. In the last section, we look at the problem of the semi-local existence of such structures.

1. Introduction

From the work of Magri [12, 13], a dynamic system admitting two compatible Hamiltonian formulations is completely integrable (under suitable conditions); i.e. the eigenvalues of the recursion operator of the bi-Hamiltonian system form a set of pairwise Poisson-commuting invariants. There are two kinds of difficulties; first, it is in general very difficult to give locally an explicit second Hamiltonian structure for a given integrable Hamiltonian system [14] even if it is theoretically always possible in the neighbourhood of a regular point of the Hamiltonian [6]; secondly we know [5–8, 10] that the global or semi-local existence of such structures implies very strong conditions which are rarely satisfied. In this paper, we define a weaker notion called a quasi-bi-Hamiltonian system (QBHS) which relaxes these two difficulties for two degrees of freedom, the only case we study. We only ask for a Hamiltonian field to be, after multiplication by an integrating factor, a Hamiltonian for a second compatible symplectic structure. In the following, when the two eigenvalues of the Nijenhuis operator are functionally independent, we give the general form of the second structure in the natural system of canonical coordinates generated by these eigenvalues (here called *Nijenhuis coordinates*) which depend only on a function A of the eigenvalues. The Hamilton–Jacobi separability in these coordinates corresponds to $A = 0$. In the last section we study the problem of the semi-local existence of QBHS; for Hamiltonians without critical points such structures always exist. In the case of the so-called 'Pfaffian QBHS' where we ask the integrating factor to have a special form, we state conditions of existence which look like the bi-Hamiltonian one [7] but are weaker.

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2. Hénon–Heiles integrable case

Let $\mathbb{R}^4$ with coordinates (q_1, q_2, p_1, p_2) be a canonical sympletic structure. Consider the Hamiltonian [9, 14, 15]:

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + Aq_1^2 + Bq_2^2) + \epsilon(q_1^2 q_2 + 2q_2^3)\,. \tag{1}$$

Without loss of generality, we can take $A = B = 1$. Starting from the well known second invariant

$$F = -\tfrac{1}{8}(3(q_1^2 + p_1^2) + \epsilon(-4q_2 p_1^2 + 4q_1 p_1 p_2 + 4q_1^2 q_2) + \epsilon^2(q_1^4 + 4q_1^2 q_2^2)) \tag{2}$$

one of us showed [14] that for the Poisson structure given by the following matrix (for $\epsilon = 1$):

$$C = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2}q_1 \\ 0 & 0 & -\frac{1}{2}q_1 & \frac{3}{4} - q_2 \\ 0 & \frac{1}{2}q_1 & 0 & \frac{1}{2}p_1 \\ \frac{1}{2}q_1 & q_2 - \frac{3}{4} & -\frac{1}{2}p_1 & 0 \end{pmatrix} \tag{3}$$

the vector field $X = \rho X_H^0$ (where $\rho^2 = \det C$ and X_H^0 is the Hamiltonian field) is also Hamiltonian with Hamiltonian F, i.e. $X = C(\cdot, F)$. Finally, we have

$$X_H^0 = C_0(\cdot, H) = \frac{1}{\rho}C(\cdot, F) \tag{4}$$

where C_0 denotes the standard Poisson structure associated with the canonical symplectic form. Denote by J the operator CC_0^{-1} linking the two structures. A straightforward calculation shows that the Nijenhuis torsion of J is equal to zero (which is the compatibility condition). The eigenvalues λ_1 and λ_2 of the Nijenhuis operator J satisfy

$$-\lambda_1\lambda_2 = \tfrac{1}{4}q_1^2 \tag{5a}$$

$$\lambda_1 + \lambda_2 = \tfrac{3}{4} - q_2 \tag{5b}$$

and are functionally independent. Moreover, $C_0(\lambda_1, \lambda_2) = 0$. So, we can find a new system of canonical coordinates $(\lambda_1, \lambda_2, p_{\lambda_1}, p_{\lambda_2})$ (that we call Nijenhuis coordinates).

To find these new canonical coordinates, we use the generating function $G(\lambda_1, \lambda_2, p_1, p_2)$ so that

$$\frac{\partial G}{\partial p_i} = -q_i \tag{6a}$$

$$\frac{\partial G}{\partial \lambda_i} = -p_{\lambda_i}\,. \tag{6b}$$

Using equation (6*a*) we can integrate and recover G:

$$G = -2\sqrt{-\lambda_1\lambda_2}\,p_1 - \left(\tfrac{3}{4} - \lambda_1 - \lambda_2\right)p_2\,. \tag{7}$$

After two derivations, we obtain

$$-p_{\lambda_1} = \frac{\partial G}{\partial \lambda_1} = \frac{-p_1}{\sqrt{\lambda_1}}\sqrt{-\lambda_2} + p_2 \tag{8a}$$

(with $\lambda_1 > 0$ and $\lambda_2 < 0$)

$$-p_{\lambda_2} = \frac{\partial G}{\partial \lambda_2} = \frac{-p_1}{\sqrt{-\lambda_2}}\sqrt{\lambda_1} + p_2 \tag{8b}$$

leading to the second part of the canonical transformation:

$$p_1 = \frac{p_{\lambda_2} - p_{\lambda_1}}{\lambda_2 - \lambda_1}\sqrt{-\lambda_1\lambda_2} \qquad p_2 = \frac{\lambda_1 p_{\lambda_1} - \lambda_2 p_{\lambda_2}}{(\lambda_2 - \lambda_1)}\,. \tag{9}$$

The relations (5a), (5b) and (9) allow us to write H and F explicitly in the Nijenhuis coordinates:

$$H = \frac{1}{8(\lambda_2 - \lambda_1)}((16\lambda_1^4 - 40\lambda_1^3 + 33\lambda_1^2 - 9\lambda_1 - 4\lambda_1 p_{\lambda_1}^2) \\ -(16\lambda_2^4 - 40\lambda_2^3 + 33\lambda_2^2 - 9\lambda_2 - 4\lambda_2 p_{\lambda_2}^2)) \tag{10}$$

and

$$F = \frac{\lambda_1\lambda_2}{8(\lambda_2 - \lambda_1)}((16\lambda_2^3 - 40\lambda_2^2 + 33\lambda_2 - 4p_{\lambda_2}^2) - (16\lambda_1^3 - 40\lambda_1^2 + 33\lambda_1 - 4p_{\lambda_1}^2))\,. \tag{11}$$

We see that H presents a Gantmacher form [3, 4], i.e.

$$H(\lambda_1, \lambda_2, p_{\lambda_1}, p_{\lambda_2}) = \frac{H_1(\lambda_1, p_{\lambda_1}) - H_2(\lambda_2, p_{\lambda_2})}{H_3(\lambda_1, p_{\lambda_1}) - H_4(\lambda_2, p_{\lambda_2})} \tag{12}$$

which implies Hamilton–Jacobi separability [3, 4].

We can see from this example that it is relatively easy to find a 'quasi-bi-Hamiltonian' formulation for the Hénon–Heiles system leading to a natural system of canonical coordinates (Nijenhuis) in which the Hamilton–Jacobi equation separates. The following step consists in trying to examine under what conditions this property is satisfied.

3. Definition and properties of quasi-bi-Hamiltonian systems

Definition 1. Let (M, ω_0, H) be a Hamiltonian system, i.e. (M, ω_0) is a symplectic manifold and $H \in \mathcal{C}^\infty(M, \mathbb{R})$. We say that it admits a quasi-bi-Hamiltonian structure if there exist:

(i) A symplectic form ω on M compatible with ω_0, i.e. the endomorphisms field J defined by $\omega_0(X, Y) = \omega(JX, Y)$ is a Nijenhuis operator, i.e. its Nijenhuis torsion N_J is equal to zero where

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y]\,. \tag{13}$$

(ii) A non-vanishing function $\rho \in \mathcal{C}^\infty(M, \mathbb{R})$ so that the 1-form $\rho i_{X_H^0}\omega$ is closed, i.e. the field ρX_H^0 is locally Hamiltonian for ω.

Given these conditions we state that the 5-tuple $(M, \omega_0, H, \omega, \rho)$ is a quasi-bi-Hamiltonian system (QBHS). We call ρ the integrating factor of the QBHS.

Remark. Note that the operator J thus defined is (as in section 2) $J = CC_0^{-1}$ where C and C_0 denote the Poisson structures associated with the symplectic forms ω and ω_0, respectively.

Definition 2. We say that a QBHS $(M, \omega_0, H, \omega, \rho)$ is exact if the field ρX_H^0 is (globally) Hamiltonian for ω.

Note that in this case if F denotes a primitive of $\rho i_{X_H}\omega$ we have $\rho X_H^0 = X_F$ (where X_F denotes the Hamiltonian vector field of the Hamiltonian F for ω) and so we call F a second Hamiltonian for X_H^0. We remark that

$$\rho X_H^0 \cdot F = X_F \cdot F = 0$$

and since $\rho \neq 0$, F is a first integral for X_H^0.

Definition 3. We say that a QBHS $(M, \omega_0, H, \omega, \rho)$ is real decomposable if its Nijenhuis operator J has the maximum number ($= \frac{1}{2}\dim M$) of distinct real eigenvalues in each point (such that J is diagonalizable).

Proposition 1 and definition. Let $(\mathbb{R}^4, \omega_0, H, \omega, \rho)$ be a QBHS (where ω_0 is the canonical symplectic form on $\mathbb{R}^4$). So it is automatically exact. Note that F is a second Hamiltonian. If H and F are functionally independent then $\rho^2/\det J$ is a first integral of the field X_H^0. We say that the QBHS is Pfaffian if $\rho = -\lambda_1\lambda_2$ where λ_1 and λ_2 denote the eigenvalues of the Nijenhuis operator J of the system.

Proof. Since H and F are functionally independent and in involution, we can find a new system (H, F, p_H, p_F) of canonical coordinates for ω_0. In these coordinates, we then have

$$X_H^0 = \frac{\partial}{\partial p_H} \qquad \text{and} \qquad X_F^0 = \frac{\partial}{\partial p_F}\,. \tag{14}$$

Moreover,

$$\begin{aligned} JX_F^0 &= CC_0^{-1}X_F^0 \\ &= CC_0^{-1}C_0\,\mathrm{d}F \\ &= C\,\mathrm{d}F \\ &= X_F \\ &= \rho X_H^0\,. \end{aligned}$$

So we obtain $JX_F^0 = \rho X_H^0$ or $J^{-1}X_H^0 = \frac{1}{\rho}X_F^0$, i.e.

$$J^{-1}\frac{\partial}{\partial p_H} = \frac{1}{\rho}\frac{\partial}{\partial p_F}\,. \tag{15}$$

Since it is easier to work with forms than Poisson brackets, we prefer to look at $J^{-1} = C_0C^{-1}$. Indeed, if we denote

$$C^{-1} = \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & \theta & \phi \\ -\beta & -\theta & 0 & \psi \\ -\gamma & -\phi & -\psi & 0 \end{pmatrix} \tag{16}$$

then these coefficients correspond directly to those of the symplectic form

$$\begin{aligned} \omega = \alpha\,\mathrm{d}H \wedge \mathrm{d}F + \beta\,\mathrm{d}H \wedge \mathrm{d}p_H + \gamma\,\mathrm{d}H \wedge \mathrm{d}p_F + \theta\,\mathrm{d}F \wedge \mathrm{d}p_H + \phi\,\mathrm{d}F \wedge \mathrm{d}p_F \\ +\psi\,\mathrm{d}p_H \wedge \mathrm{d}p_F\,. \end{aligned} \tag{17}$$

With these notations we have

$$J^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & \theta & \phi \\ -\beta & -\theta & 0 & \psi \\ -\gamma & -\phi & -\psi & 0 \end{pmatrix} = \begin{pmatrix} \beta & \theta & 0 & -\psi \\ \gamma & \phi & \psi & 0 \\ 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & \theta & \phi \end{pmatrix}. \tag{18}$$

Comparing (15) and (18), we obtain $\beta = \psi = 0$ and $\theta = 1/\rho$. So

$$J^{-1} = \begin{pmatrix} 0 & 1/\rho & 0 & 0 \\ \gamma & \phi & 0 & 0 \\ 0 & \alpha & 0 & \gamma \\ -\alpha & 0 & 1/\rho & \phi \end{pmatrix} \tag{19}$$

leading to $\gamma^2/\rho^2 = \det J^{-1} = 1/\det J$ or

$$\rho^2 = \gamma^2 \det J\,. \tag{20}$$

We recover

$$\omega = \alpha\,\mathrm{d}H \wedge \mathrm{d}F + \gamma\,\mathrm{d}H \wedge \mathrm{d}p_F + \frac{1}{\rho}\,\mathrm{d}F \wedge \mathrm{d}p_H + \phi\,\mathrm{d}F \wedge \mathrm{d}p_F\,. \tag{21}$$

Writing $\mathrm{d}\omega = 0$, and by looking at the terms $\mathrm{d}p_H \wedge \mathrm{d}H \wedge \mathrm{d}p_F$, we obtain $\frac{\partial\gamma}{\partial p_H} = 0$, i.e. $X_H^0 \cdot \gamma = 0$. Then γ is a first integral for X_H^0. Consequently, if λ_1 and λ_2 denote the eigenvalues of J, $\rho^2/(\lambda_1\lambda_2)^2$ is a first integral for X_H^0. The particular case (called Pfaffian) where $\rho = -\lambda_1\lambda_2$ corresponds to $\gamma = 1$.

Remark. In this calculation, we do not use the compatibility condition—if we use the compatibility we can show that $\gamma = \gamma(H, F)$.

Proposition 2 and definition. Let ω_0 and ω be two compatible symplectic forms on $\mathbb{R}^4$. Let λ_1 and λ_2 be the eigenvalues of their Nijenhuis operator J.

(i) λ_1 and λ_2 are in involution for the two structures. If λ_1 and λ_2 are functionally independent, we can complete (λ_1, λ_2) in a system of canonical coordinates $(\lambda_1, \lambda_2, p_{\lambda_1}, p_{\lambda_2})$ that we call Nijenhuis coordinates.

(ii) In these coordinates, if J is diagonalizable, the symplectic forms ω_0 and ω take the following form:

$$\begin{aligned}\omega_0 &= \mathrm{d}\lambda_1 \wedge \mathrm{d}p_{\lambda_1} + \mathrm{d}\lambda_2 \wedge \mathrm{d}p_{\lambda_2}\\ \omega &= \frac{1}{\lambda_1}\mathrm{d}\lambda_1 \wedge \mathrm{d}p_{\lambda_1} + \frac{1}{\lambda_2}\,\mathrm{d}\lambda_2 \wedge \mathrm{d}p_{\lambda_2} + A(\lambda_1, \lambda_2)\,\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2\,.\end{aligned} \tag{22}$$

(iii) If $A = 0$, and if $(\mathbb{R}^4, \omega_0, H, \omega, \rho = -\lambda_1\lambda_2)$ is a Pfaffian real decomposable QBHS with a second Hamiltonian F then H and F present the following Gantmacher form:

$$H = \frac{H_1(\lambda_1, p_{\lambda_1}) - H_2(\lambda_2, p_{\lambda_2})}{\lambda_1 - \lambda_2} \tag{23}$$

$$F = \frac{-\lambda_2 H_1(\lambda_1, p_{\lambda_1}) + \lambda_1 H_2(\lambda_2, p_{\lambda_2})}{\lambda_1 - \lambda_2} \tag{24}$$

so we have Hamilton–Jacobi separability.

Proof. We first recall the classical result about compatible structures (see [11] for example). If ω_0 and ω are two compatible symplectic structures with a diagonalizable Nijenhuis operator, the eigenspaces E_{λ_1} and E_{λ_2} form ω_0-orthogonal integrable distributions and λ_1 (respectively λ_2) is a first integral for E_{λ_2} (respectively E_{λ_1}).

(i) It is a classical result of bi-Hamiltonian system theory which only uses the compatibility of the two forms. Here we can recall a direct proof in this simple case: note that if f is a first integral of E_{λ_1} we have $\forall X \in E_{\lambda_1}, \omega_0(X_f^0, X) = -\mathrm{d}f(X) = 0$. Hence

$$X_f^0 \in E_{\lambda_1}^{\perp} = E_{\lambda_2}.$$

Then, $\{\lambda_1, \lambda_2\}_0 = -X_{\lambda_2}^0 \cdot \lambda_1 = 0$ because $X_{\lambda_2}^0 \in E\lambda_2$.

(ii) We write

$$\omega_0 = \mathrm{d}\lambda_1 \wedge \mathrm{d}p_{\lambda_1} + \mathrm{d}\lambda_2 \wedge \mathrm{d}p_{\lambda_2}$$

and

$$\begin{aligned}\omega = {} & A\,\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2 + B\,\mathrm{d}\lambda_1 \wedge \mathrm{d}p_{\lambda_1} + C\,\mathrm{d}\lambda_1 \wedge \mathrm{d}p_{\lambda_2} + D\,\mathrm{d}\lambda_2 \wedge \mathrm{d}p_{\lambda_1} + E\,\mathrm{d}\lambda_2 \wedge \mathrm{d}p_{\lambda_2}\\ & + F\,\mathrm{d}p_{\lambda_1} \wedge \mathrm{d}p_{\lambda_2}\,.\end{aligned}$$

We have $X_{\lambda_1}^0 = \partial/\partial p_{\lambda_1}, X_{\lambda_2}^0 = \partial/\partial p_{\lambda_2}$ and

$$-\mathrm{d}\lambda_1 = i_{\partial/\partial_{p_{\lambda_1}}}\omega_0 = i_{J_{\partial/\partial_{p_{\lambda_1}}}}\omega = i_{\lambda_1 \partial/\partial_{p_{\lambda_1}}}\omega = -\lambda B\,\mathrm{d}\lambda_1 - \lambda_1 D\,\mathrm{d}\lambda_2 + \lambda_1 F\,\mathrm{d}p_{\lambda_2}$$

yielding $B = 1/\lambda_1$, $D = F = 0$. Identically, we obtain by using $\partial/\partial_{p_{\lambda_2}}$

$$E = \frac{1}{\lambda_1} \qquad C = F = 0.$$

Finally, using $\mathrm{d}\omega = 0$, we conclude that $A = A(\lambda_1, \lambda_2)$ proving the required result (22).

(iii) Writing $X_H^0 = \frac{1}{\rho} X_F = \frac{-1}{\lambda_1\lambda_2} X_F$, we obtain four relations between the partial derivatives of H and F in the coordinates $(\lambda_1, \lambda_2, p_{\lambda_1}, p_{\lambda_2})$. A straightforward integration yields the required result.

4. The problem of the semi-local existence of QBHS

We know from [5–8, 10] that the global or even the semi-local existence of a real decomposable bi-Hamiltonian system implies very strong conditions which are rarely satisfied. The next question is to ask if this would also be the case for QBHS. Since the definition of QBHS being weaker than the classical one, we could expect that the conditions would be less restrictive. Indeed, we shall see in the following that for two degrees of freedom there is always a solution (but not necessary Pfaffian) for non-critical Hamiltonians. For Pfaffian QBHS we shall state a result which looks like the classical bi-Hamiltonian one [7], but with a weaker condition for existence.

By 'semi-local existence' we mean the following (see [1, 2] for more details). If (M, ω_0, H) is a completely integrable Hamiltonian system with compact level sets we know from the Arnold–Liouville theorem [2] that their connected components are tori and that each of these components admits an open neighbourhood with a symplectomorphism from this open on $U \times \mathbb{T}^2$ (where U is an open set of $\mathbb{R}^2$) endowed with a canonical symplectic form such that the first integrals providing the complete integrability of the system (particularly the Hamiltonian) depend only on the coordinates on U (the action coordinates). The problem here is to know if it is possible to have a quasi-bi-Hamiltonian structure for this system over the whole of such a neighbourhood of the so-called Liouville's torus. By the Arnold–Liouville theorem it is equivalent to studying this existence on $U \times \mathbb{T}^2$ with coordinates (x, y, θ, ϕ), $\omega_0 = \mathrm{d}x \wedge \mathrm{d}\theta + \mathrm{d}y \wedge \mathrm{d}\phi$ and $H = H(x, y)$.

Definitions. We say that a function is basic if it is basic for the fibration $\pi : U \times \mathbb{T}^2 \to U$, i.e. depends only on action coordinates x, y. A basic function is said to be non-degenerate if its Hessian matrix has a maximum rank on a dense open set of U.

By the Arnold–Liouville theorem, the Hamiltonian H is basic. We can show that if it is non-degenerate only the first integrals of X_H^0 are basic functions.

We shall first note a remark of F J Turiel [16] as follows.

Proposition 3. If H has no critical point, there always exists a neighbourhood $V \times \mathbb{T}^2 (V \subset U)$ of each Liouville's torus which admits a real decomposable QBHS.

Proof. Let $\omega_1 = \lambda(x)\,\mathrm{d}x \wedge \mathrm{d}\theta + \mu(y)\,\mathrm{d}y \wedge \mathrm{d}\phi$. This form is symplectic if λ, μ are non-vanishing functions and the 1-form on U:

$$i_{X_H^0}\omega_1 = \lambda(x)\frac{\partial H}{\partial x}\,\mathrm{d}x + \mu(y)\frac{\partial H}{\partial y}\,\mathrm{d}y$$

admits a (basic) integrating factor on some $V \subset U$.

Remark.

(i) This argument is false for opens of $\mathbb{R}^n$, $n > 2$.
(ii) This solution is not necessary Pfaffian.

For the Pfaffian case we can now state the following result.

Proposition 4. If a Pfaffian real decomposable quasi-bi-Hamiltonian structure exists for the Hamiltonian system $(U \times \mathbb{T}^2, \omega_0, H)$ with H non-degenerate, with a basic integrating factor then there exist local coordinates (u_1, u_2) on U so that the functions x, y, H split as

$$\begin{aligned} x &= x_1(u_1) + x_2(u_2) \\ y &= y_1(u_1) + y_2(u_2) \\ H &= \frac{H_1(u_1) + H_2(u_2)}{H_3(u_1) + H_4(u_2)}\,. \end{aligned} \tag{25}$$

Remark. This result looks like the bi-Hamiltonian one but the condition on H, x, y is weaker. For example, all the Hamiltonians of the type $H(x, y) = f(x)g(y)$ satisfy these conditions. In particular, the counterexample $H(x, y) = x(y^2 + 1)$ of [6].

Proof. We write

$$\omega = A\,\mathrm{d}x \wedge \mathrm{d}y + B\,\mathrm{d}x \wedge \mathrm{d}\theta + C\,\mathrm{d}x \wedge \mathrm{d}\phi + D\,\mathrm{d}y \wedge \mathrm{d}\theta + E\,\mathrm{d}y \wedge \mathrm{d}\phi + F\,\mathrm{d}\theta \wedge \mathrm{d}\phi.$$

Looking at the terms in $\mathrm{d}\theta \wedge \mathrm{d}\phi$ in the equation $\mathrm{d}(\rho i_{X_H}\omega) = 0$, we get

$$\rho\left(\frac{\partial F}{\partial \phi}\frac{\partial H}{\partial y} + \frac{\partial F}{\partial \theta}\frac{\partial H}{\partial x}\right) = 0$$

i.e. $(\rho \neq 0) X_H \cdot F = 0$. Then, because H is non-degenerate, F is a basic function. Now, if we look at the terms in $\mathrm{d}x \wedge \mathrm{d}\theta$ in the equation $\mathrm{d}(\rho i_{X_H}\omega) = 0$, we obtain

$$\rho\frac{\partial B}{\partial \theta}\frac{\partial H}{\partial x} + \rho\frac{\partial C}{\partial \theta}\frac{\partial H}{\partial y} = \frac{\partial \rho}{\partial x}\frac{\partial H}{\partial y} + F\frac{\partial^2 H}{\partial x \partial y}\,.$$

But $\mathrm{d}\omega = 0$ so in particular we have $\frac{\partial C}{\partial \theta} = \frac{\partial B}{\partial \phi} + \frac{\partial F}{\partial x}$ and so using the last two equations we obtain that $X_H \cdot B$ is basic. Then,

$$X_H\left(\frac{\partial B}{\partial \theta}\right) = \frac{\partial}{\partial \theta}(X_H \cdot B) = 0$$

so $\frac{\partial B}{\partial \theta}$ is basic.

Hence B is an affine function in θ, ϕ with basic coefficients and thus does not depend on θ, ϕ because it is a global function on $U \times \mathbb{T}^2$. Hence B is basic. Similarly, the other coefficients of ω are basic. But, in these conditions we obtain that F is constant because ω is closed. Finally, because $\rho i_{X_H}\omega$ is closed, we get that the product of F by each of the coefficients of the Hessian matrix of H is zero and so $F = 0$ (H is non-degenerate).

In conclusion, J projects on U, and with the same arguments as in the bi-Hamiltonian case [6, 7] we see that its projection is a diagonalizable Nijenhuis operator $\bar{J}$ and that there exist local coordinates (u_1, u_2) on $\mathbb{R}^2$ so that

$$\bar{J} = \lambda_1(u_1)\,\mathrm{d}u_1 \otimes \frac{\partial}{\partial u_1} + \lambda_2(u_2)\,\mathrm{d}u_2 \otimes \frac{\partial}{\partial u_2}.$$

We still have $\mathrm{d}x \circ \bar{J}^{-1}$ and $\mathrm{d}y \circ \bar{J}^{-1}$ closed but the difference with the bi-Hamiltonian case is that $\rho\,\mathrm{d}H \circ \bar{J}^{-1}$ is closed but $\mathrm{d}H \circ \bar{J}^{-1}$ is not.

The hypothesis that the QBHS is Pfaffian, i.e. $\rho = -\lambda_1\lambda_2$, gives

$$\mathrm{d}\left(-\lambda_1\lambda_2\left(\frac{1}{\lambda_1}\frac{\partial H}{\partial u_1}\,\mathrm{d}u_1 + \frac{1}{\lambda_2}\frac{\partial H}{\partial u_2}\,\mathrm{d}u_2\right)\right) = 0$$

yielding, by a straightforward calculation

$$\frac{\partial^2}{\partial u_1 \partial u_2}((\lambda_2 - \lambda_1)H) = 0$$

i.e.

$$(\lambda_2(u_2) - \lambda_1(u_1))H = H_1(u_1) + H_2(u_2)$$

hence the result.

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